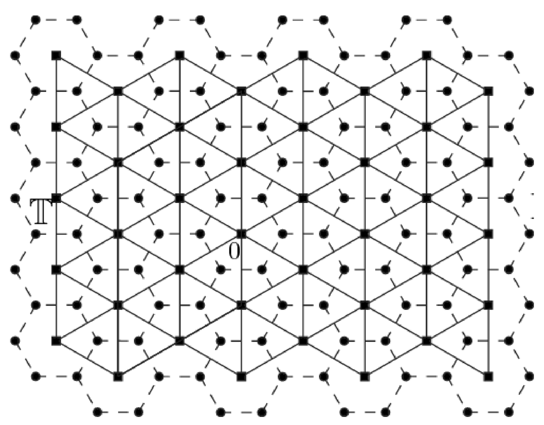
Sharp asymptotics of arm probabilities in critical planar percolation

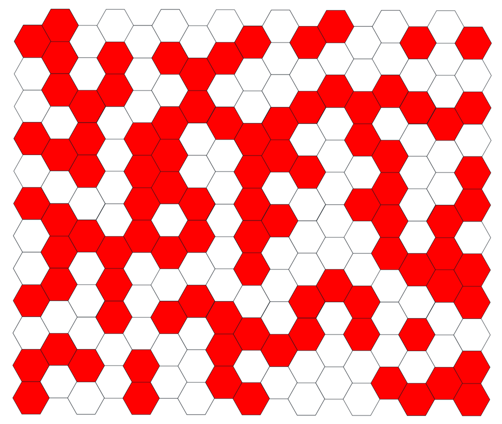
Hang Du†, Yifan Gao*, Xinyi Li†, Zijie Zhuang§

†Peking University, *City University of Hong Kong, §University of Pennsylvania

Mathematical Settings

- We consider the critical planar percolation on the triangular lattice $\mathbb{T}$: each site of $\mathbb{T}$ is open independently with probability $1/2$.
- For illustration, we paint hexagons in the dual lattice $\mathbb{T}^*$ instead, and use two different colors to represent openness and closedness.





Arm Events

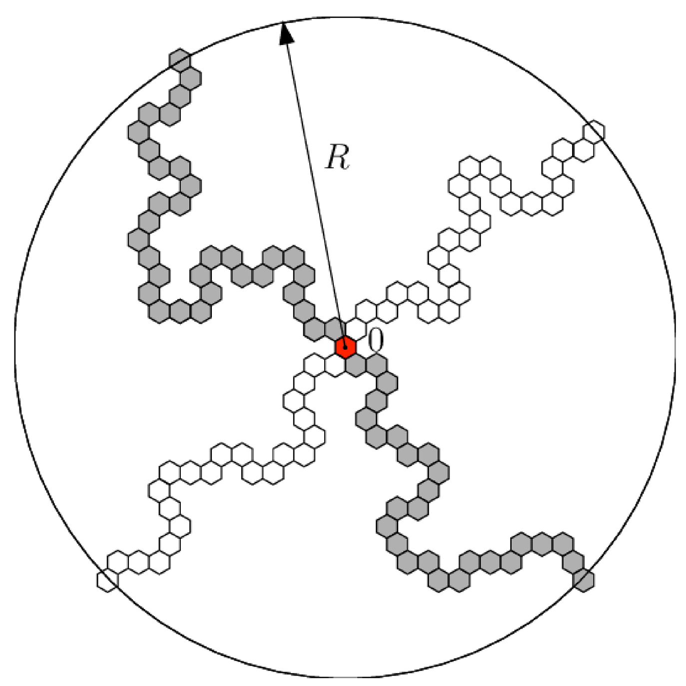
- An **arm** is a self-avoiding path of nearest-neighbor hexagons of the same color.
- C_R: circle of radius R centered at the origin; $A(r,R)$: the annulus with boundaries C_r, C_R.

Definition (Arm events)**.** *The half-plane j-arm event $\mathcal{B}_j(r,R)$:*

$$\{\exists j \text{ disjoint arms of alternating colors not leaving } \mathbb{H}, \text{ and each of them connects } C_r \text{ to } C_R\}.$$

The whole-plane j-arm event $\mathcal{P}_j(r,R)$:

$$\{\exists j \text{ disjoint arms connecting } C_r \text{ to } C_R \text{ with alternating colors}\}.$$



An example $\mathcal{P}_4(1,R)$, which also satisfies $\mathcal{B}_2(1,R)$.

Previous results

- The probabilities of Arm events are central objects of interest for the study percolation. The most classical results is the followings:

Theorem (Simirnov-Werner' 01)**.**
Half-plane *exponent: For any $j \geq 1$,*

$$\mathbb{P}[\mathcal{B}_j(r,R)] = R^{-j(j+1)/6+o(1)}.$$

Whole-plane *exponent: For any $j \geq 2$,*

$$\mathbb{P}[\mathcal{P}_j(r,R)] = R^{-(j^2-1)/12+o(1)}.$$

Theorem (Lawler-Schramm-Werner '01)**.**

$$\mathbb{P}[\mathcal{P}_1(r,R)] = R^{-5/48+o(1)}.$$

- All these results leave an $o(1)$ factor in the exponent. Oded Schramm asked for up-to-constant estimates in ICM 2006.
- $\mathbb{P}[\mathcal{B}_2(1,R)] \asymp R^{-1}$, $\mathbb{P}[\mathcal{B}_3(1,R)] \asymp R^{-2}$ and $\mathbb{P}[\mathcal{P}_5(1,R)] \asymp R^{-2}$ can be derived from elementary arguments. However, improvements for other cases are much more difficult.

Technical Input

- A **power-law rate for convergence** of the exploration process to SLE_6: consider a Jordan set Ω with $a,b \in \partial\Omega$.
  - Let γ be the cordal SLE_6 in Ω from a to b.
  - For $\eta > 0$, with suitable discretization $(\Omega_\eta, a_\eta, b_\eta)$ by $\eta\mathbb{T}^*$, let γ_η be the exploration process from a_η to b_η.
  - Given open $U \subset \Omega$, such that $a \notin \partial U$ and $b \in \partial U$, let T_η (resp. T) be the first time that γ_η (resp. γ) enters U_η (resp. U).

Theorem (Binder-Richards '21)**.**
Under mild assumptions, $\exists u > 0$ s.t. $\forall \eta > 0$, there is a coupling $\mathbf{P}$ of γ_η and γ such that

$$\mathbf{P}\left[d\left(\gamma_\eta|_{[0,T_\eta]}, \gamma|_{[0,T]}\right) > \eta^u\right] < O(\eta^u),$$

where d is the up-to-reparametrization metric between two curves.

Our Main Results

- We are now able to give sharp asymptotics for arm probabilities.

Theorem (D.-G.-L.-Z. '22+)**.**
In the half-plane case, for any $j \geq 1$, $r \geq r_0(j)$, $\exists C, c > 0$ s.t.

$$\mathbb{P}[\mathcal{B}_j(r,R)] = CR^{-j(j+1)/6}\big(1 + O(R^{-c})\big).$$

In the whole-plane case, for any $j \geq 2$, $r \geq r_0'(j)$, $\exists C' > 0$, s.t.

$$\mathbb{P}[\mathcal{P}_j(r,R)] = C'R^{-(j^2-1)/12}\big(1 + o(1)\big).$$

In particular, one can take $r_0(1) = 1$ for $j = 1,2,3$ and $r_0'(1) = 1$ for $j = 2,3,4,5,6$.

One More Result

- We also obtain the following **super strong separation lemma** in the half-plane, which solves a conjecture in [Garban-Pete-Schramm'13]:

Theorem (D.-G.-L.-Z.'22+)**.**
$\forall j \geq 1$, $\exists K, c > 0$ such that whenever $R_0 \geq R > Kr$, conditioned on the event $\mathcal{B}_j(r,R_0)$ together with **any realization** *of the color configuration outside C_R, the land-points on C_r of interfaces crossing $A(r,R)$ are* **well separated** *with probability at least c.*

References

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