Characterizing the limiting Potts measure on locally $\mathtt{T}_d$-like graphs: Local weak limits and strong phase coexistence

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1.Potts measure on locally $\mathtt{T}_d$-like graphs

- The q**-state Potts measure** on G with inverse temperature β and external field h is defined as a Gibbs measure on $[q]^{V(G)}$:

$$\mu^G_{\beta,h}(\sigma) = \frac{1}{\mathcal{Z}_{\beta,h}(G)} \exp\left(\beta \sum_{i\sim j} \mathbf{1}\{\sigma(i)=\sigma(j)\} + h \sum_i \mathbf{1}\{\sigma(i)=1\}\right).$$

- We focus on the Potts measures on a sequence of graphs that **locally converges to the infinite** d**-regular tree** $\mathtt{T}_d$ in the *Benjamini-Schramm* sense (i.e. a typical neighborhood is a d-regular tree).

2.Background: Bethe prediction and BP fixed points

- For any sequence of locally $\mathtt{T}_d$-like graphs $\{G_n\}$, the **Bethe prediction** suggests that the limiting free energy density is given by

$$\lim_{n\to\infty} \frac{1}{|V_n|} \log \mathcal{Z}_{\beta,h}(G_n) = \sup \Psi(\nu),$$

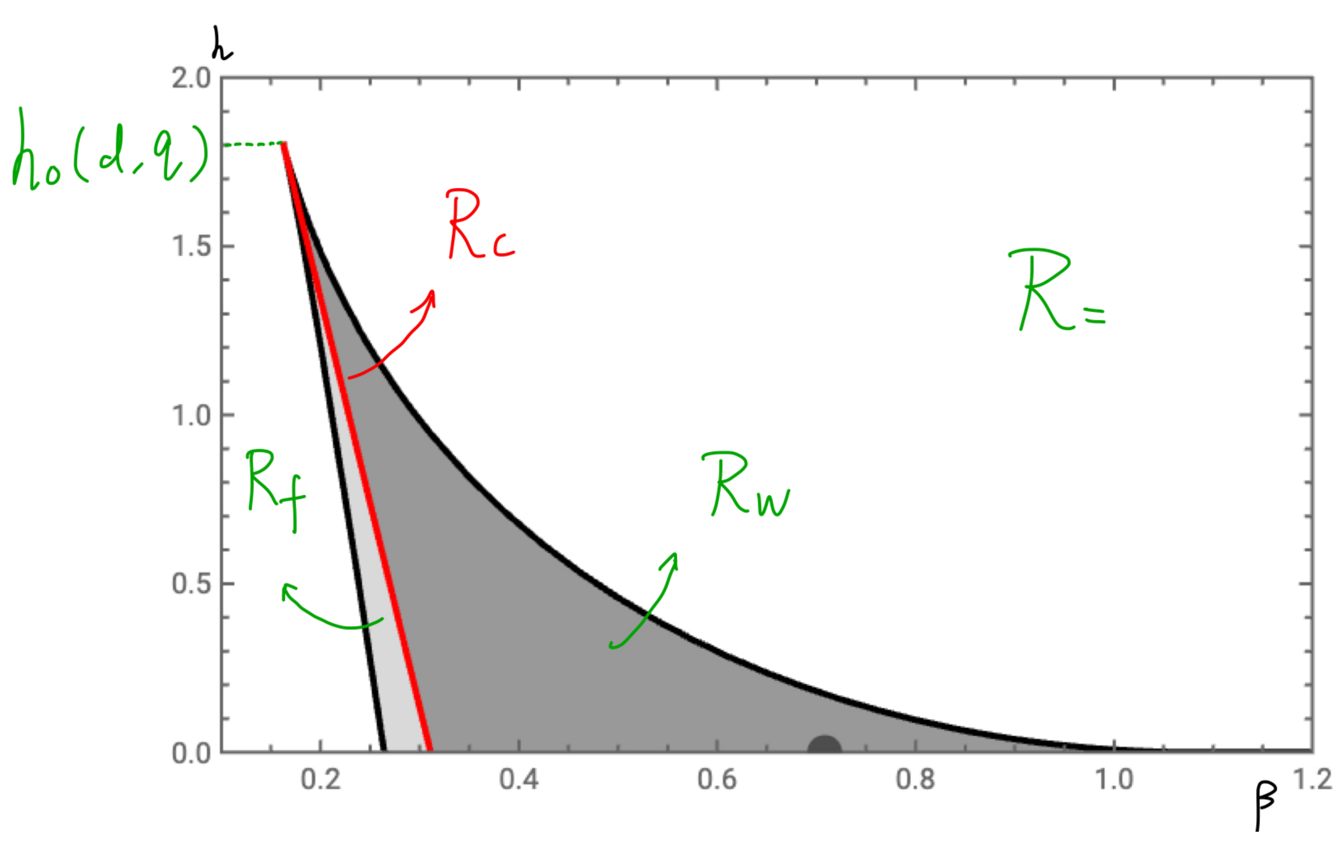
where $\Psi : \mathcal{P}([q]) \to \mathbb{R}$ is the **Bethe functional** and the supremum is taken over all probability measures on $[q]$.
- [DMSS14] resolve the Bethe variational problem by showing

$$\sup \Psi(\nu) = \max\{\Psi(\nu^{\mathrm{f}}_{\beta,h}), \Psi(\nu^{\mathrm{w}}_{\beta,h})\},$$

where $\nu^{\mathrm{f}}_{\beta,h}$ and $\nu^{\mathrm{w}}_{\beta,h}$ are the **belief-propagation fixed points** corresponding to the free and wired boundary conditions of Potts measure on $\mathtt{T}_d$.
- The bethe prediction has been proved in [DMSS14, CvdH25].

3.Background: The Potts phase diagram

- The Bethe variational principle yields the **phase diagram** of Potts measures on $\mathtt{T}_d$ (and hence on locally $\mathtt{T}_d$-like graphs).
- For the Ising case ($q=2$), it turns out that for all $(\beta,h)\in\mathbb{R}^2_+$, $\nu^{\mathrm{f}}_{\beta,h}=\nu^{\mathrm{w}}_{\beta,h}$, and thus the analysis is relatively easier [DM10].
- However, the non-uniqueness regime for general Potts measures ($q\ge 3$) is two-dimensional, due to the presence of two distinct phase transitions: **uniqueness-nonuniquenss, ordered- disordered**, making the analysis much more challenging.



Phase diagram of the Potts measures on $\mathbb{T}_d$ when $d=25, q=45$.

- $\mathcal{R}_=$ is the uniqueness regime, $\nu^{\mathrm{f}}_{\beta,h}=\nu^{\mathrm{w}}_{\beta,h}$
- On $\mathcal{R}_\dagger$, $\Psi(\nu^\dagger_{\beta,h}) > \Psi(\nu^\ddagger_{\beta,h})$; On $\mathcal{R}_c$, $\Psi(\nu^{\mathrm{f}}_{\beta,h}) = \Psi(\nu^{\mathrm{w}}_{\beta,h})$.

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4.Background: Potts measure local weak convergence

- Physicists conjecture that, in the large n limit, the local spin profile around a random vertex converges to a mixture of pure states induced by the dominant measures $\nu^\dagger_{\beta,h}, \dagger\in\{\mathrm{f},\mathrm{w}\}$. This is known as **local weak convergence** (of Gibbs measures).
- For $(\beta,h)\in\mathbb{R}^2\setminus\mathcal{R}_c$, the dominant measure is unique, and hence $\mu^{G_n}_{\beta,h}$ converges to a single $\mu^\dagger_{\beta,h}, \dagger\in\{\mathrm{f},\mathrm{w}\}$; whereas for $(\beta,h)\in\mathcal{R}_c$, there are two dominant measures $\nu^{\mathrm{f}}_{\beta,h}, \nu^{\mathrm{w}}_{\beta,h}$, and the local weak limit should be a mixture of $\mu^{\mathrm{f}}_{\beta,h}, \mu^{\mathrm{w}}_{\beta,h}$.
- The non-critical regime was fully resolved in [BDS25], assuming the Bethe prediction. In the critical regime $\mathcal{R}_c$, [HJP23, BDS25] made progress for the case q is *large* and d is *even*.

5. Results: local weak limit and phase coexistence

- We resolve the local weak limit conjecture in $\mathcal{R}_c$ for **expander graphs** (certain well-connected graphs) for general $d,q\ge 3$.

Theorem 1. For any $d,q\ge 3$, any $(\beta,h)\in\mathcal{R}_c$ and any sequence of locally $\mathtt{T}_d$-like expander graphs, any locally weakly convergent subsequence of $\{\mu^{G_n}_{\beta,h}\}$ converges **locally weakly in probability** to a mixture of $\mu^{\mathrm{f}}_{\beta,h}$ and $\mu^{\mathrm{w}}_{\beta,h}$.

- We further show that the mixture weight can be arbitrary, thereby confirming the **strong phase coexistence** prediction.

Theorem 2. For any $d,q\ge 3$, any $(\beta,h)\in\mathcal{R}_c$ and any $\alpha\in[0,1]$, there exists a sequence of graphs G_n such that $G_n \xrightarrow{\text{loc}} \mathbb{T}_d$ and that

$$\mu^{G_n}_{\beta,h} \xrightarrow{\text{lwcp}} \alpha\mu^{\mathrm{f}}_{\beta,h} + (1-\alpha)\mu^{\mathrm{w}}_{\beta,h}.$$

6.Proof ingredients

- *Basic strategy*: Following [BDS23], we utilize the **Edward-Sokal coupling** to transition to analyzing the **random cluster measure**, where we can exploit *monotonicity*.
- *Key ingredient*: A generalized local version of the **rank-2 approximation** of random cluster partition function established and developed in [BBC23, CvdH25].
- *Main technical input*: A *large deviation type estimate* for the **cluster sizes** of the **FK-Ising percolation** on locally $\mathtt{T}_d$-like expander graphs inspired from [KLS20].

References

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