Central limit theorem for log-concave measures
Course project for 18.156

Hang Du

May 12, 2025

Abstract

A series of works by Anttila, Ball, Perissinaki, and Klartag establish that for a high-dimensional isotropic log-concave measure μ, its projection onto a random direction θ is, with high probability, well-approximated by the standard Gaussian. In this note, we survey the essential steps of the proof, with a focus on the high-level ideas and techniques.

1 Introduction

Let μ be a probability measure on $\mathbb{R}^n$ with density $\mu(x)$. We say that μ is *log-concave* if

$$\mu(x)^\lambda \mu(y)^{1-\lambda} \le \mu(\lambda x + (1-\lambda)y)\,, \quad \forall x, y \in \mathbb{R}^n\,,\ 0 < \lambda < 1\,.$$

Classical examples of log-concave measures include the Gaussian measure and the uniform measure on a convex body. Moreover, by the Brunn–Minkowski inequality, any marginal of a log-concave measure is also log-concave.

Log-concave measures play a central role in high-dimensional geometry, connecting the field to various branches of mathematics such as functional analysis, probability theory, and theoretical computer science. Consequently, the study of log-concave measures has become a fundamental topic in high-dimensional geometry and has attracted significant attention over the past few decades.

An intriguing aspect of their connection to probability theory is the generalized central limit theorem, which concerns how well a typical one-dimensional marginal of a high-dimensional log-concave measure approximates the Gaussian distribution. Since the mean and variance of the marginal may vary with the direction, it is natural to introduce the following normalizing condition, known as *isotropy*. For a log-concave measure μ on $\mathbb{R}^n$, we say that μ is *isotropic* if for a random vector $X = (X_1, \ldots, X_n) \sim \mu$,

$$\mathbb{E}[X_i] = 0\,, \quad \mathbb{E}[X_i X_j] = \delta_{ij}\,, \quad \forall\, 1 \le i, j \le n\,.$$

We refer to this as a normalizing condition because every non-degenerate measure μ can be transformed into an isotropic measure via an affine transformation.

Regarding notational conventions, we always assume that the dimension $n \to \infty$, and use the standard asymptotic notations: $f_n = o(g_n)$, $f_n = \omega(g_n)$, $f_n = O(g_n)$, and $f_n = \Omega(g_n)$ mean, respectively, that $f_n/g_n \to 0$, $f_n/g_n \to \infty$, $\limsup f_n/g_n < \infty$, and $\liminf f_n/g_n > 0$ as $n \to$

∞. Moreover, we use w.h.p. (short for "with high probability") to denote that the corresponding probability is $1 - o(1)$ as $n \to \infty$.

The main result that this note focuses on is that the marginal of a high-dimensional isotropic log-concave measure along a random direction is w.h.p. close to the standard Gaussian distribution.

Theorem 1. *Let μ be an isotropic log-concave measure on $\mathbb{R}^n$. For any $\theta \in \mathbb{S}^{n-1}$, denote μ_θ for the distribution of $\langle X, \theta \rangle$ where $X \sim \mu$. Then as $n \to \infty$, for θ uniformly sampled from $\mathbb{S}^{n-1}$, w.h.p. it holds*

$$\mathrm{TV}(\mu_\theta, \mathcal{N}(0,1)) = o(1) \,. \tag{1}$$

We state Theorem 1 in its simplest form, but we remark that it has many generalizations. For example, for a subspace E of R^n, denote by μ_E the projection of μ onto E. One may expect that for a typical choice of E of dimension M not too large, w.h.p. μ_E is close to $\mathcal{N}(0, I_M)$. This is indeed true, as shown in [Kla07], provided that $M = o(\log n / \log\log n)$. Additionally, a strong (and nearly sharp) quantitative version of (1), closely related to the KLS conjecture, has been obtained only very recently in [FK22].

In this note, we survey the key steps in the proof of Theorem 1, following [ABP03] and [Kla07]. Both papers made significant contributions toward establishing Theorem 1. Roughly speaking, [ABP03] reduces the central limit theorem problem to showing that the measure μ satisfies a "thin shell" condition, meaning that the mass of μ concentrates in a small neighborhood of the sphere in R^n with radius $\sqrt{n}$, while [Kla07] verifies that every isotropic log-concave measure satisfies this thin shell condition. Both steps may appear surprising at first glance, and the goal of this note is to explain the intuition behind them and the techniques involved in their proofs. Most of the arguments in this note are borrowed from [ABP03] and [Kla07]. We also refer to [Fre21] for a short and accessible proof of Theorem 1.

2 Reduction to the thin shell condition

Definition 2. We say μ satisfies the thin shell condition, if

$$\mu\big[|X| = (1 + o(1))\sqrt{N}\big] = 1 - o(1) \,. \tag{2}$$

Let $G \sim \mathcal{N}(0,1)$ be a standard Gaussian variable. The main goal of this section is to show the following proposition:

Proposition 3. *Assume that μ is an isotropic log-concave measure on $\mathbb{R}^n$ that satisfies the thin shell condition (2). then for any fixed interval $I = [a,b] \subset \mathbb{R}$, as $n \to \infty$, for a random direction $\theta \in \mathbb{S}^{n-1}$, it holds w.h.p. that*

$$\mu_\theta[I] = \mathbb{P}[G \in I] + o(1) \,.$$

Proposition 3 provides a weak version of the central limit theorem under the assumption of the thin shell condition. For simplicity, we state here only a non-quantitative version of Proposition 3, although the corresponding probability estimates can in fact be made very strong. It is not hard to believe that by applying a more quantitative version of Proposition 3 and combining it with standard ε-net and continuity arguments, one can deduce the full central limit theorem as stated in Theorem 1 under the thin shell condition. Therefore, in what follows, we only sketch the proof of Proposition 3, and refer interested readers to [ABP03, Section 1] for the remaining details.

In what follows, we denote by $\mathbb{P}_\theta$ and $\mathbb{E}_\theta$ the probability and expectation with respect to θ sampled from the uniform measure on the sphere S^{n-1}. The importance of the thin shell condition is illustrated by the following simple lemma.

Lemma 4. *Fix $I = [a, b] \subset \mathbb{R}$. As $n \to \infty$, it holds that for any measure μ on $\mathbb{R}^n$,*

$$\mathbb{E}_\theta\big[\mu_\theta[I]\big] = (1 + o(1))\mathbb{E}_\mu\big[\mathbb{P}[|X|/\sqrt{n} \cdot G \in I]\big]\,.$$

Therefore, if the thin shell condition is satisfied, then $\mathbb{E}_\theta\big[\mu_\theta[I]\big] = \mathbb{P}[G \in I] + o(1)$.

Proof. By definition we have

$$\mathbb{E}_\theta\big[\mu_\theta[I]\big] = \mathbb{E}_\theta\big[\mathbb{E}_\mu[\mathbf{1}\{\langle X, \theta\rangle \in I]\big] = \mathbb{E}_\mu[\mathbb{P}_\theta[\langle X, \theta\rangle \in I]]\,.$$

Notice that for θ uniformly sampled from $\mathbb{S}^{n-1}$, $\langle X, \theta\rangle$ has the same distribution as $|X|\theta_1$, where θ_1 is the first coordinate of $\theta \in \mathbb{S}^{n-1}$. It is a standard fact that as $n \to \infty$, the distribution of θ_1 is $o(1)$-TV close to $\mathcal{N}(0, 1/\sqrt{N})$.[1] This concludes the proof. $\square$

In light of Lemma 4, Proposition 3 follows upon showing that $\mu_\theta[I]$ tightly concentrates around its expectation. For $\theta \in \mathbb{S}^{n-1}$ and $t \in \mathbb{R}$, we let

$$M_\theta(\mu, t) = \mu[X : \langle X, \theta\rangle \le t]\,. \tag{3}$$

Since for $I = [a, b]$, we have $\mu_\theta[I] = M_\theta(\mu, b) - M_\theta(\mu, a)$, it suffices to prove the following concentration property of $M_\theta(\mu, t)$ over θ uniformly sampled from $\mathbb{S}^{n-1}$.

Proposition 5. *There exists a universal constant $c > 0$, such that for any $t \in \mathbb{R}$ and $\delta > 0$,*

$$\mathbb{P}_\theta\left[\Big|M_\theta(\mu, t) - \mathbb{E}_\theta\big[M_\theta(\mu, t)\big]\Big| \ge \delta\right] \le 2\exp(-c\delta^2 n)\,. \tag{4}$$

Proof. Fix $t \in \mathbb{R}$. It suffices to verify that the function $\theta \mapsto M_\theta(\mu, t)$ is $O(1)$-Lipschitz on $\mathbb{S}^{n-1}$, and then (4) follows from the standard concentration inequality on the sphere.

For $\theta_1, \theta_2 \in \mathbb{S}^{n-1}$ such that $|\theta_1 - \theta_2| \le 0.1$, we have

$$|M_{\theta_1}(\mu, t) - M_{\theta_2}(\mu, t)| \le \mu\big[\{\langle X, \theta_1\rangle \le t\}\Delta\{\langle X, \theta_2\rangle \le t\}\big]\,. \tag{5}$$

Let q be the projection of μ on $\mathrm{span}(\theta_1, \theta_2)$, which we identify with $\mathbb{R}^2$. By (5) we have

$$|M_{\theta_1}(\mu, t) - M_{\theta_2}(\mu, t)| \le \int_\square q(x, y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

where the region of integration is the blue region depicted as in Figure 1 ($\beta = \arccos\langle\theta_1, \theta_2\rangle$). Since that q is an isotropic log-concave measure on $\mathbb{R}^2$, we have that $q(x, y) \le C\exp(-c|x| - c|y|)$ for some universal constants $c, C > 0$. Using this estimate, we readily conclude that

$$|M_{\theta_1}(\mu, t) - M_{\theta_2}(\mu, t)| \le 2Ce^{-|t|}|\theta_1 - \theta_2|\,,$$

and thus $\theta \mapsto M_\theta(\mu, t)$ is $O(1)$-Lipschitz, as desired. This completes the proof. $\square$

[1] One may think of sampling a uniform $\theta \in \mathbb{S}^{n-1}$ by first sampling a gaussian vector $x \sim \mathcal{N}(0, I_n)$, and then let $\theta = X/|X|$. The result then follows from the law of large number that w.h.p. $|X| = (1 + o(1))\sqrt{n}$.

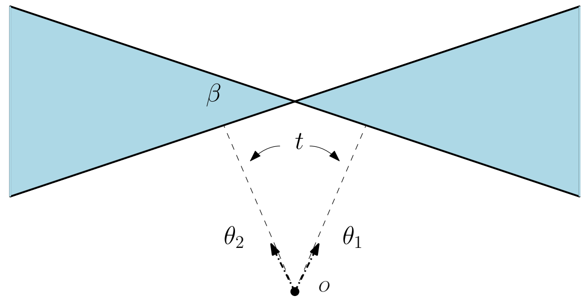

Figure 1: The integrand region.

3 Verifying the thin shell condition

[Kla07] proves the surprising result that any isotropic log-concave measure satisfies the thin shell condition, thereby establishing the central limit theorem based on the reduction in [ABP03]. In this section, we sketch the high-level approach in [Kla07] for verifying the thin shell condition for general isotropic log-concave measures.

We begin with a dimension reduction lemma enabled by a random projection. For integers $n \geq k \geq 1$, let $G_{n,k}$ denote the (n,k)-Grassmannian, and let $\sigma_{n,k}$ denote the uniform distribution (Haar measure) on $G_{n,k}$. For any $E \in G_{n,k}$, we write π_E for the projection from R^n onto E.

Lemma 6 (Johnson-Lindenstrauss dimension reduction lemma)**.** *Let X be a vector in $\mathbb{R}^n$. For $k = \omega(1)$ and $E \sim \sigma_{n,k}$, it holds w.h.p. that*

$$|\pi_E X| = (1 + o(1)) \cdot \sqrt{\frac{k}{n}} |X|$$

Intuitively, by projecting onto a random subspace, the norm of X becomes almost evenly distributed across all directions. To ensure sufficient cancellation and achieve strong concentration, the subspace dimension k must be large enough, which motivates the condition $k = \omega(1)$. The proof is a straightforward application of concentration inequalities, and we refer to [JL84] for details.

In the remainder of this section, we always set $k = (\log n)^{1/2}$.[2] In light of Lemma 6, to verify that μ satisfies the thin shell condition, it suffices to show that w.h.p. over $X \sim \mu$ and $E \sim G_{n,k}$, we have $|\pi_E X| = (1 + o(1))\sqrt{k}$.

At first glance, such a reduction may seem useless, as k is still growing with n. However, in what follows, we will define certain "good" subspaces onto which the projection of μ enjoys desirable properties. We will argue that a random subspace $E \in G_{n,k}$ is w.h.p. good. This allows us to restrict our attention to a good subspace E and to study the projected measure μ_E. Notice that μ_E possesses certain desirable properties by the definition of good subspaces, whereas a priori there is no reason to assume that μ itself satisfies such properties.

We now proceed to define the good subspaces.

Definition 7. A subspace $E \in G_{n,k}$ is said to be good, if the following holds

$$\sup_{\theta \in E, |\theta|=1} M_\theta(\mu, f) - \inf_{\theta \in E, |\theta|=1} M_\theta(\mu, f) \leq n^{-0.1}, \quad \forall t \in \mathbb{R}. \tag{6}$$

[2] Indeed, any choice satisfying $1 \ll k \ll \log n$ would work equally well for the proof.

Proposition 8. *For $E \in G_{n,k}$, E is good with high probability.*

Proof. We will not give the full proof of Proposition 8 but only provide a heuristic justification. In section 2 we show that for any fixed t, the function $\theta \mapsto M_\theta(\mu, t)$ tightly concentrates around its expectation, denoted as $M(t)$. Precisely, by applying the tail estimate (4), we see that

$$\mathbb{P}_\theta \left[|M_\theta(\mu,t) - M(t)| > \tfrac{1}{2}n^{-0.1}\right] \leq \exp(-\Omega(n^{0.8}))\,. \tag{7}$$

Moreover, similar as mentioned in Section 2, by continuity arguments we may extends (7) to

$$\mathbb{P}_\theta \left[|M_\theta(\mu,t) - M(t)| > \tfrac{1}{2}n^{-0.1} \text{ for some } t \in \mathbb{R}\right] \leq \exp(-\Omega(n^{0.8}))\,. \tag{8}$$

We call a point $\theta \in \mathbb{S}^{n-1}$ to be good, if

$$|M_\theta(\mu,t) - M(t)| \leq \tfrac{1}{2}n^{-0.1}\,, \quad \forall \in \mathbb{R}\,.$$

Then (8) implies that at least an $1 - \exp(-\Omega(n^{-0.8})$ portion of points $\theta \in \mathbb{S}^{n-1}$ are good.

On the other hand, we notice that for $E \in G_{n,k}$, if any $\theta \in E \cap \mathbb{S}^{n-1}$ is good, then E itself is good. When E is randomly sampled from $\sigma_{n,k}$, we may think of testing all points $\theta \in E \cap \mathbb{S}^{d-1}$ for being good as testing $\exp(O(k))$ many random points on $\mathbb{S}^{n-1}$ for being good. Since $\exp(O(k)) = \exp(O((\log n)^{1/2}) \ll n^{0.1}$, in view of the union bound, we may expect that w.h.p. over $E \sim \sigma_{n,k}$, any $\theta \in E \cap \mathbb{S}^{n-1}$ is good. This concludes a heuristic proof of Proposition 8, and we refer to [Kla07, Lemma 3.2] for details. □

In what follows we fix a good subspace E. For notational convenience, we identify E with $\mathbb{R}^k$ and let $\nu = \mu_E$. Then, ν_E is an isotropic log-concave measure on $\mathbb{R}^k$ with the property that

$$\sup_{\theta \in \mathbb{S}^{k-1}} M_\theta(\nu,t) - \inf_{\theta \in \mathbb{S}^{k-1}} M_\theta(\nu,t) \leq e^{-100k}\,. \tag{9}$$

Notice that condition (9) suggests a strong spherical symmetry of ν, which closely aligns with our goal of showing that $\nu[|X| = (1+o(1))\sqrt{k}] = 1 - o(1)$.

Inspired by the nice connection between projections and Fourier transforms, a natural approach is to first use (9) to show that $\hat{\nu}$—the Fourier transform of ν—is approximately spherically symmetric, and then apply the Fourier inversion formula to deduce that the same holds for ν, and thus concludes the desired result. However, this approach cannot be directly applied, as one must carefully control the error terms. In particular, the errors become non-negligible when estimating $\hat{\nu}(\xi)$ for ξ with large modulus.

To handle this issue, we introduce a mollifier. Consider $\lambda = \nu * g$, where $g(x) = (2\pi)^{-k/2}e^{-|x|^2/2}$ is the standard Gaussian density. Then, $\hat{\lambda}(\xi) = \hat{\nu}(\xi) \cdot e^{-|\xi|^2/2}$, and we can easily conclude that $\hat{\lambda}$ is nearly spherical-symmetric. This yields the following proposition.

Proposition 9. *Assume ν satisfies (9). Then for $\lambda = \nu * g$, it holds that*

$$\sup_{\theta \in \mathbb{S}^{n-1}} \lambda(r\theta) - \inf_{\theta \in \mathbb{S}^{k-1}} \lambda(r\theta) \leq e^{-20n}\,, \quad \forall r \geq 0\,. \tag{10}$$

Proof. For $\xi \in \mathbb{R}^k$, we write $r = |\xi| \in [0, \infty)$ and $\theta = r^{-1}\xi \in \mathbb{S}^{k-1}$. By definition we have

$$\hat{\nu}(\xi) = \int_{\mathbb{R}^k} e^{-ix\cdot\xi}\nu(x)\,\mathrm{d}x = \int_{\mathbb{R}} e^{-irt}\nu_\theta(t)\,\mathrm{d}\,.$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}M_\theta(\nu,t)=\nu_\theta(t)$, using integrate by part formula we get (the boundary terms vanish)

$$\hat{\nu}(\xi)=ir\int_{\mathbb{R}}e^{-irt}M_\theta(\nu,t)\,\mathrm{d}t\,.$$

Therefore, for any $\xi_1,\xi_2\in\mathbb{R}^k$ with $|\xi_1|=|\xi_2|=r$, denoting $\theta_i=r^{-1}\xi_i, i=1,2$, we have

$$|\hat{\nu}(\theta_1)-\hat{\nu}(\theta_2)|\leq r\Big|\int_{\mathbb{R}}e^{-irt}(M_{\theta_1}\big(\nu,t(-M_{\theta_2}(\nu,t)\big)\,\mathrm{d}t\Big|\,.$$

Since ν is an isotropic log-concave measure, it is an elementary fact that $M_\theta(\nu,t)\leq 2e^{-|t|/10}$ for any $t\in\mathbb{R}$ and $\theta\in\mathbb{S}^{k-1}$ (see, e.g. [Kla07, Lemma 2.2]). Therefore, combining with (9) we have

$$|\hat{\nu}(\xi_1)-\hat{\nu}(\xi_2)|\leq r\cdot\Big(e^{-100k}\cdot 400k+4\int_{200k}^{\infty}e^{-t/10}\,\mathrm{d}t\Big)\leq re^{-20k}\,.$$

Since $\hat{\lambda}(\xi)=\hat{\nu}(\xi)\cdot e^{-|\xi|^2/2}$, we have for any $\xi_1,\xi_2\in\mathbb{R}^k$ with $|\xi_1|=|\xi_2|=r$,

$$|\lambda(\xi_1)-\lambda(\xi_2)|\leq re^{-r^2/2}\cdot e^{-20k}\,.$$

Now, for any x_1,x_2 with $|x_1|=|x_2|=r$, let $U\in\mathrm{O}(k)$ such that $Ux_1=x_2$, then $x_1\cdot\xi=x_2\cdot U\xi,\forall\xi\in\mathbb{R}^k$. Using the Fourier inversion formula, we have

$$\begin{aligned}|\lambda(x_1)-\lambda(x_2)|&=(2\pi)^{-k/2}\Big|\int_{\mathbb{R}^k}e^{-x_1\cdot\xi}\,\mathrm{d}\xi-\int_{\mathbb{R}^k}e^{ix_2\cdot\xi}\hat{\lambda}(\xi)\,\mathrm{d}\xi\Big|\\&\leq(2\pi)^{-k/2}\Big|\int_{\mathbb{R}^k}e^{ix_1\cdot\xi}(\hat{\lambda}(\xi)-\hat{\lambda}(U\xi))\,\mathrm{d}\xi\Big|\\&\leq(2\pi)^{-k/2}\int_{\mathbb{R}^k}|\xi|e^{-|\xi|^2/2}\cdot e^{-20k}\,\mathrm{d}\xi\,,\end{aligned}$$

which is at most e^{-20k}, as desired. This concludes the proof. □

Note that λ can be viewed as the distribution of $X+Y$, where $X\sim\nu,Y\sim\mathcal{N}(0,I_k)$ are independent. It is easy to see that w.h.p. it holds

$$|X+Y|^2=|X|^2+|Y|^2+2\langle X,Y\rangle=|X|^2+(1+o(1))k+o\big(\sqrt{k}|X|\big)\,.$$

Therefore, in order to show that $\nu[|X|=(1+o(1))\sqrt{k}]=1-o(1)$, it suffices to prove that $\lambda[|X|=(1+o(1))\sqrt{2k}]=1-o(1)$. The following lemma further reduces our goal to merely show that $|X|$ is well-concentrated under λ.

Lemma 10. *Let λ be an isotropic log-concave measure on $\mathbb{R}^k$ such that $\mathbb{E}_\lambda[|X|^2]=2k$. If there exists r_0 such that $\lambda[|X|=(1+o(1))r_0]=1-o(1)$, then $\lambda[|X|=(1+o(1))\sqrt{k}]=1-o(1)$.*

Lemma 10 is totally not surprising. The tricky point is that for a general probability distribution, the expectation may not reflect its typical behavior. However, it turns out that for log-concave measures there is no such issue. We refer to [Kla07, Lemma 4.6] for the proof of Lemma 10.

We now proceed to verify that $|X|$ concentrates tightly around some value under μ. While (10) somewhat suggests that this is indeed the case, the proof is not so straightforward. We start with some intuitions that motivate the later proof. For simplicity, let us first assume that λ is actually spherically symmetric, i.e. for some function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$, it holds that $\lambda(x) = f(|x|), \forall x \in \mathbb{R}^n$. It is easy to check that f is also log-concave. We have for any $I \subset \mathbb{R}$,

$$\lambda[|X| \in I] = \frac{\int_I r^{k-1} f(r)\,\mathrm{d}r}{\int_0^\infty r^{k-1} f(r)\,\mathrm{d}r}\,.$$

We write $r^{k-1} f(r) = \exp((k-1)\log r + \log f(r))$. Note that

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}((k-1)\log r + \log f(r)) = -\frac{k-1}{r^2} + (\log f(r))'' < 0\,,$$

the function $r \mapsto (k-1)\log r + \log f(r)$ is concave, and thus it has a unique maximizer r_0 in $[0, \infty)$. Moreover, since as $k \to \infty$, the function decays rapidly near r_0, in view of Laplace's method (a.k.a. the saddle point method), we expect that the main contribution of the integral of $r^{k-1} f(r)$ comes from a small interval around r_0. This heuristically concludes that $\nu[|X| = (1+o(1))r_0] = 1-o(1)$.

In general, we fix a small constant $\varepsilon > 0$, and we wish to show that as $k \to \infty$, there exists $r_0 > 0$ such that for $A_{r_0,\varepsilon} = \{(1-\varepsilon)r_0 \leq |X| \leq (1+\varepsilon)r_0\}$, it holds $\lambda[A_{r_0,\varepsilon}] = 1 - o(1)$. By the polar coordinate transform, we have

$$\lambda[A_{r_0,\varepsilon}] = \int_{\mathbb{S}^{k-1}} \int_{(1-\varepsilon)r_0}^{(1+\varepsilon)r_0} r^{k-1} \lambda(r\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,,$$

whereas

$$= \int_{\mathbb{S}^{k-1}} \int_0^\infty r^{k-1} \lambda(r\theta)\,\mathrm{d}r\,\mathrm{d}\theta = 1\,.$$

If we could pick r_0 such that for each direction $\theta \in \mathbb{S}^{k-1}$, the most contribution of the integral

$$\int_0^\infty r^{k-1} \lambda(r\theta)\,\mathrm{d}r \tag{11}$$

comes from $r \in ((1-\varepsilon)r_0, (1+\varepsilon)r_0)$, then the desired result follows. However, as hinted earlier, the main contribution of the integral (11) should come from a small interval around the maximizer of $r \mapsto r^{k-1}\lambda(r\theta)$. Therefore, it remains to show that the maximizers do not different from each other too much across different θ.

Precisely, we introduce the following definition.

Definition 11. For $\theta \in \mathbb{S}^{n-1}$, let $r_k(\theta)$ be the (unique) maximizer of the function $r \mapsto r^{k-1}\lambda(r\theta)$.

The following lemma is a straightforward conclusion of the Laplace method, and we refer to [Kla07, Lemma 4.5] for details.

Lemma 12. *For any $\varepsilon > 0$ and $\theta \in \mathbb{S}^{n-1}$, it holds that*

$$\int_{(1-\varepsilon/2)r_k(\theta)}^{(1+\varepsilon/2)r_k(\theta)} r^{k-1} \lambda(r\theta)\,\mathrm{d}r \geq (1 - o(1)) \int_0^\infty r^{k-1} \lambda(r\theta)\,\mathrm{d}r\,.$$

Finally, we show that $\theta \mapsto r_k(\theta)$ is almost constant on $\mathbb{S}^{k-1}$.

Lemma 13. *For two log-concave C^2 functions $f, g : [0,\infty) \to [0,\infty)$, let $r_k(f)$ (respectively, $r_k(g)$) be the maximizer of $r \mapsto r^{k-1}f(r)$ (respectively $r \mapsto r^{k-1}g(r)$). Assume that $|f(t)-g(t)| \le e^{-5k}\min\{f(0), g(0)\}$ for all $t \in \mathbb{R}$, then it holds that $(1-e^{-k})r_k(f) \le r_k(g) \le (1+e^{-k})r_k(f)$. In particular, assuming (10), it holds that*

$$\sup_{\theta \in \mathbb{S}^{k-1}} r_k(\theta) \le (1+e^{-k}) \inf_{\theta \in \mathbb{S}^{k-1}} r_k\theta\,. \tag{12}$$

Proof. By symmetry we only need to show that $\theta_k(g) \ge (1-e^{-k})\theta_k(f)$. Without loss of generality, we may assume $\theta_k(f) = 1$. Denote $f_0 = \log f, g_0 = \log g$, then both f_0 and g_0 are concave. It is straightforward to check that $\theta_k(g)$ is the unique solution to the equation $tg_0'(t) = -(k-1)$, and our assumption implies $f_0'(1) = -(k-1)$. Moreover, we have $f_0'(t) \ge -(n-1), \forall t \in [0,1]$, and thus $f(t) \ge e^{-k}f(0), \forall t \in [0,1]$. Hence, the condition on f, g indicates that $|f(t)/g(t) - 1| \le e^{-4k}$, and thus

$$-2e^{-4k} < \log(1-e^{-4k}) \le f_0(t) - g_0(t) \le \log(1+e^{-4k}) \le e^{-4k}\,, \quad \forall t \in [0,1] \tag{13}$$

We claim that

$$g_0'(t) \ge f_0'(t+e^{-2k}) - 4e^{-2k}\,, \quad \forall 0 \le t \le 1-e^{-2k}\,.$$

If this is not true, we have for some $t_0 \in [0, 1-e^{-2k}]$, we have for any $t \in [t_0, t_0+e^{-2k}]$,

$$g_0'(t) \le g_0'(t_0) \le f_0'(t_0+e^{-2k}) - 4e^{-2k} \le f_0'(t) - 4e^{-2k}\,,$$

where in the first and the last inequality we used the fact that g_0, f_0 are concave. This implies that for $h_0(t) = f_0(t) - g_0(t)$, $h_0'(t) \ge 4e^{-2k}$ for any $t \in [t_0, t_0+e^{-2k}]$, and thus

$$h_0(t_0+e^{-2n}) - h_0(t_0) \ge 4e^{-2k} \cdot e^{-2k} = 4e^{-4k}\,.$$

This contradicts (13) and thus the claim holds.

The claim implies that for any $s \in [0, 1-e^{-n})$ with $f_0'(s) < 0$,

$$sg_0'(s) > (1-e^{-k})(f'(1) - 4e^{-2k}) = -(1-e^{-k})(k-1+4e^{-2k}) > -(n-1)\,,$$

and this means $\theta_k(g) \ge 1-e^{-k}$, as desired. $\square$

Finally we complete the proof that $|X|$ is concentrated under λ, thereby verifying the thin shell condition. (12) implies that for any $\varepsilon > 0$, as $k \to \infty$ there exists r_0 such that $(1-\varepsilon)r_0 \le (1-\varepsilon/2)r_k(\theta) < (1+\varepsilon/2)r_k(\theta) < (1+\varepsilon)r_0$. Therefore, by the polar coordinate transform and Lemma 12, for $A_{r_0,\varepsilon} = \{(1-\varepsilon)r_0 \le |X| \le (1+\varepsilon)r_0\}$ it holds that

$$\begin{aligned}
\lambda[A_{r_0,\varepsilon}] &\ge \int_{\mathbb{S}^{k-1}} \int_{(1-\varepsilon/2)r_k(\theta}^{(1+\varepsilon/2)r_k(\theta)} r^{k-1}\lambda(r\theta)\,\mathrm{d}r\,\mathrm{d}\theta \\
&\ge (1-o(1)) \int_{\mathbb{S}^{k-1}} \int_0^\infty r^{k-1}\lambda(r\theta)\,\mathrm{d}r\,\mathrm{d}\theta \\
&= (1-o(1)) \int_{\mathbb{R}^k} \lambda(x)\,\mathrm{d}x = 1-o(1)\,.
\end{aligned}$$

This proves that $\lambda[|X| \in ((1-\varepsilon)r_0, (1+\varepsilon)r_0)] = 1-o(1)$. Since this is true for any $\varepsilon > 0$, the desired result follows.

References

[ABP03] Milla Anttila, Keith Ball, and Irini Perissinaki. The central limit problem for convex bodies. *Transactions of the American Mathematical Society*, 355(12):4723–4735, 2003.

[FK22] Xiao Fang and Yuta Koike. Sharp high-dimensional central limit theorems for log-concave distributions. *arXiv preprint arXiv:2207.14536*, 2022.

[Fre21] Daniel J. Fresen. A simplified proof of clt for convex bodies. *Proceedings of the American Mathematical Society*, 149(1):345–354, 2021.

[JL84] William B. Johnson and Joram Lindenstrauss. Extensions of lipschitz mappings into a hilbert space. In *Contemporary Mathematics*, volume 26, pages 189–206. American Mathematical Society, 1984.

[Kla07] Boáz Klartag. A central limit theorem for convex sets. *Inventiones Mathematicae*, 168(1):91–131, 2007.